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The blockage of gravity and capillary waves by longer waves and currents

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Surface waves superimposed upon a larger-scale flow are blocked at the points where the group velocities balance the convection by the larger-scale flow. Two types of blockage, capillary and gravity, are investigated by using a new multiple-scale technique, in which the short waves are treated linearly and the underlying larger-scale flows are assumed steady but can have a considerably curved surface and uniform vorticity. The technique first provides a uniformly valid second-order ordinary differential equation, from which a consistent uniform asymptotic solution can readily be obtained by using a treatment suggested by the result of Smith (1975) who described the phenomenon of gravity blockage in an unsteady current with finite depth.

The corresponding WKBJ solution is also derived as a consistent asymptotic expansion of the uniform solution, which is valid at points away from the blockage point. This solution is obviously represented by a linear combination of the incident and reflected waves, and their amplitudes take explicit forms so that it can be shown that even with a significantly varied effective gravity g' and constant vorticity, wave action will remain conserved for each wave. Furthermore, from the relative amplitudes of the incident and reflected waves, we clearly demonstrate that the action fluxes carried by the two waves towards and away from the blockage point are equal within the present approximation.

The blockage of gravity–capillary waves can occur at the forward slopes of a finite-amplitude dominant wave as suggested by Phillips (1981). The results show that the blocked waves will be reflected as extremely short capillaries and then dissipated rapidly by viscosity. Therefore, for a fixed dominant wave, all wavelets shorter than a limiting wavelength will be suppressed by this process. The minimum wavelengths coexisting with the long waves of various wavelengths and slopes are estimated.

1. Introduction

The growth and decay of gravity–capillary waves within a variety of sea surface conditions have received increasing attention recently owing to the application of remote sensing techniques. The principles of operation of various sensing devices and the characteristics of the sea surface that influence the return signals are reviewed in Phillips (1988). There are many mechanisms by which energy of the short gravity–capillary waves is produced, dissipated and redistributed. Among these, the interaction between long and short waves can cause modulation in both amplitude and wavelength of the short waves (Longuet-Higgins 1987 and Henyey *et al.* 1988

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have recently re-examined this problem). Another relevant phenomenon is the modulation and blockage of short waves by the surface velocity distribution produced by underlying internal waves, which create a special surface-roughness pattern with periodic bands as has been observed both visually and with radars. An even more dramatic change, which occurs in certain circumstances, is the blockage and subsequent suppression of the short gravity–capillary waves by a dominant gravity wave near its crests, suggested first by Phillips (1981).

In an asymptotic analysis in a system of orthogonal curvilinear coordinates, Phillips (1981) has shown that the effects of a finite-amplitude long wave on short wave can be represented by an effective gravitational acceleration g', which is the component of the true acceleration g normal to the moving surface of the long wave plus the centripetal acceleration arising from a curved surface of the long wave. With the use of g' instead of g, Phillips found that the modulation of a linear short wave by a steady long, finite-amplitude dominant wave is precisely analogous with that by a steady and slowly varying current. Therefore, in a frame of reference moving with the long wave, the observed frequency of the short wave

$$n_0 = \sigma + kU = \text{constant}, \tag{1.1}$$

with

$$\sigma = (g'k + \gamma k^3)^{\frac{1}{2}}, \tag{1.2}$$

where U is the tangential surface velocity representing the resultant of the long-wave orbital velocity and the velocity arising from the translation of the frame of reference. Expression (1.2) represents the relation between the local intrinsic frequency σ and the wavenumber k of the short wave, which, except for the use of g', is exactly analogous to the dispersion relation for a level mean surface. Now if the long and short waves propagate in the same direction, then U will be negative while k and σ are positive, and since the magnitude of the convective velocity U is in general greater than the phase velocity of the short wave, from (1.1) the apparent frequency n_0 will be negative. From these results, Phillips (1981) concluded that 'with a dominant wave of sufficient steepness (though not necessarily breaking) short dispersive components are unable to propagate past the dominant wave crest'.

The kinematics of this process are illustrated in figure 1, where the two solution points labelled A and B are the intersections of the straight line

$$m = n_0 - Uk \tag{1.3}$$

with the curve (1.2). The solution A represents a gravity wave while the solution B corresponds to a capillary wave. However, as $|U|$ and therefore the slope of the straight line (1.3) gradually decreases along the long-wave profile, points A and B in figure 1 will approach each other. Then for sufficiently weak U, which may occur near the crest of a steep long wave, the solution points A and B will eventually coalesce. If U further reduces, there will be no real solution for k, so that the wave pattern is confined to one side of the coalescence point.

On the other hand, since the group velocity C_g is equal to the slope of the tangent of the curve $\sigma = (g'k + \gamma k^3)^{\frac{1}{2}}$ versus k, figure 1 also indicates that the characteristic velocity $U + C_g$ is negative for point A and positive for B; the point where A and B coalesce then corresponds to the blockage point where the group velocity C_g exactly balances the convection U. Therefore, it is possible for the gravity wave A to be reflected as the capillary wave B (or vice versa).

The above situation reminds us of the phenomenon of gravity blockage by a larger-scale current (see, for example, Peregrine 1976 and Mei 1983). Such a phenomenon

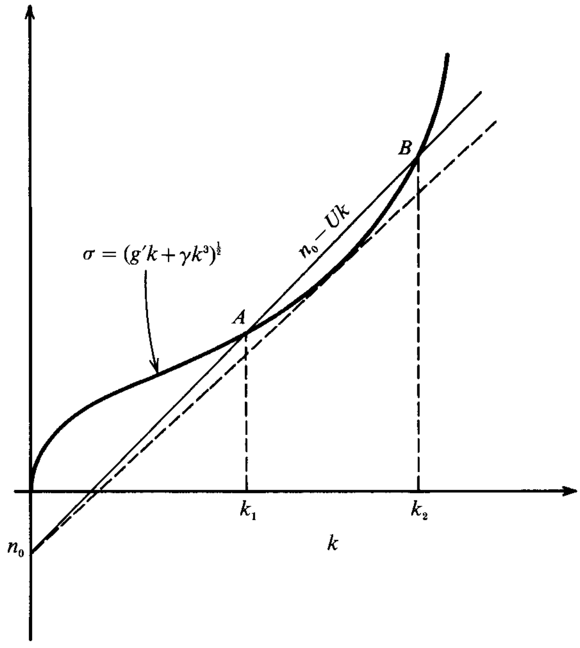

FIGURE 1. Solutions of the kinematical conservation equation for given n_0. The variations of k_1 and k_2 can be traced by a change of U and therefore the slope of the straight line $m = n_0 - Uk$.

was first recognized by Unna (1942), though a detailed analysis of its dynamics was not available until the work by Smith (1975). In careful expansions of the equations and solution, Smith has successfully obtained a uniformly valid asymptotic solution of the original boundary-value problem, which indicates that the singularities at the blockage points (or caustics) predicted by the ray solution do not exist and the blocked waves will indeed be reflected at a different wavelength at those points. Smith's (1975) work was concerned with blockage of gravity waves on an irrotational flow (though it is not restricted to straight caustics or to steady currents), and we wish to enquire whether an analogous phenomenon will also occur with capillary waves and in other conditions.

This question will be resolved here by using a new approach which includes the effects of surface tension and a curved mean surface, and yields a uniformly valid second-order ordinary differential equation for the surface displacement of the short waves. One significant outcome of the present approach is that the uniform asymptotic solution of this equation can easily be found and justified in a treatment motivated by the results of Smith (1975). This solution and the corresponding WKBJ solution are in an explicit form, so that they will be applied to clarify some important features of the modulation theory, including the principle of action conservation in the situations when g' varies significantly. Another important application of the solutions is to clarify the above-mentioned mechanism for suppression of freely travelling gravity–capillary waves.

After the general theory has been established, in §6 the similarity between the gravity and capillary blockages will be discussed and, for comparison, the solutions of the gravity blockage will be derived separately. Lastly, it will be seen that the present solutions can easily be extended to the case when waves propagate on a flow of uniform vorticity.

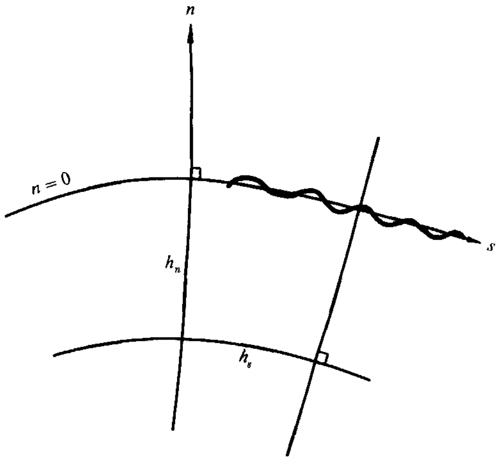

FIGURE 2. Definition sketch.

2. General formulation of short-wave fields

For the sake of definiteness, we here consider a train of short waves superimposed upon a steady, finite-amplitude long wave; the results are, however, valid for waves propagating on any steady larger-scale irrotational currents (or even rotational currents if a slight modification is made).

In a frame of reference moving with the long wave, the flow induced by the long wave becomes steady, and we may choose a set of orthogonal curvilinear coordinates (s, n) in which the undisturbed water surface of the long wave corresponds to a coordinate line. The particular coordinate system chosen here (figure 2) is the one used by Longuet-Higgins (1953). In this system the lines n = constant are parallel curves and $n = 0$ corresponds to the undisturbed surface. Thus the scale factor h_n in the n-direction is independent of the position, and on the other hand, if at the surface we set the scale factor in the s-direction $h_s = 1$, the variation of h_s in the n-direction has the simple relation

$$h_s = 1 - \frac{n}{R}, \tag{2.1}$$

where R is the radius of curvature of the undisturbed surface of the long wave, and has the order of magnitude L^2/A; A and L denote respectively the amplitude and wavelength of the long wave. This coordinate system will certainly produce singularities of the field equation, but since it is not the present purpose to solve the complete flow field, the system used here is especially suitable for the later treatment of asymptotic approximations.

In this two-dimensional coordinate system, the Laplace equation for the velocity potential ϕ of the short wave takes the form

$$\nabla^2\phi = \frac{1}{h_s^2}\frac{\partial^2\phi}{\partial s^2} + \frac{\partial^2\phi}{\partial n^2} + \frac{1}{h_s}\frac{\partial h_s}{\partial n}\frac{\partial\phi}{\partial n} = 0, \tag{2.2}$$

or

$$\frac{\partial^2\phi}{\partial s^2} + h_s\frac{\partial}{\partial n}\left(h_s\frac{\partial\phi}{\partial n}\right) = 0. \tag{2.3}$$

Accordingly, we may define a new independent variable r, given by

$$h_s \frac{\partial}{\partial n} = \frac{\partial}{\partial r}, \tag{2.4}$$

and (2.3) transforms into

$$\frac{\partial^2 \phi}{\partial s^2} + \frac{\partial^2 \phi}{\partial r^2} = 0.$$

Therefore we have the general solution

$$\phi = f(s - \mathrm{i}r) + h(s + \mathrm{i}r),$$

where f and h as the functions of a single independent variable can be any twice differentiable functions. In our case f and h will oscillate in the s-direction, at least in the regions where the ray theory is valid. Thus, one of f and h will grow and another will decay exponentially in the r (or n)-direction. It is not difficult to see that if the phase of oscillation increases in the positive s-direction, for instance $f = \mathrm{e}^{\mathrm{i}(s - \mathrm{i}r)}$, then $f(s - \mathrm{i}r)$ will represent the solution satisfying the deep-water boundary condition when $r \to -\infty$. Since, according to figure 1, both incident and reflected waves have phases propagating in the same direction, it is possible without loss of generality to choose their phases increasing in the positive s-direction. Consequently, we have definitely

$$\phi(s, r) = f(s - \mathrm{i}r), \tag{2.5}$$

from which we immediately obtain

$$\frac{\partial \phi}{\partial s} = \mathrm{i}\frac{\partial \phi}{\partial r}$$

or

$$\frac{\partial \phi}{\partial s} = \mathrm{i}h_s \frac{\partial \phi}{\partial n}$$

by virtue of (2.4). Recall that $h_s = 1$ at $n = 0$. Thus we finally have

$$\frac{\partial \phi}{\partial s} = \mathrm{i}\frac{\partial \phi}{\partial n} \quad \text{at} \quad n = 0. \tag{2.6}$$

The derivation of relation (2.6) is exact within the deep-water assumption; the latter, however, does not impose any serious restriction in application of the theory, because the waves under consideration propagate on a larger-scale flow, therefore almost always being on deep water. The above relation will later be applied to combine the two surface boundary conditions into one equation.

The exact kinematical boundary condition in the curvilinear coordinates is

$$\frac{\partial \eta}{\partial t} + \left(U + \frac{1}{h_s}\frac{\partial \phi}{\partial s}\right)\left(\frac{1}{h_s}\frac{\partial \eta}{\partial s}\right) = W + \frac{\partial \phi}{\partial n} \quad \text{at} \quad n = \eta, \tag{2.7}$$

expressing the requirement of zero flow through free surface. In (2.7), η is the surface displacement of the short wave in the n-direction as shown in figure 3; U and W represent the velocity components of the long-wave field in the s- and n-directions respectively. We now assume that the slope of the short wave is very small, so that after substitution of (2.1) and Taylor series expansions about $n = 0$, equation (2.7) can be approximated by

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial s} - \frac{\partial \phi}{\partial n} = \eta\frac{\partial W}{\partial n} \quad \text{at} \quad n = 0, \tag{2.8}$$

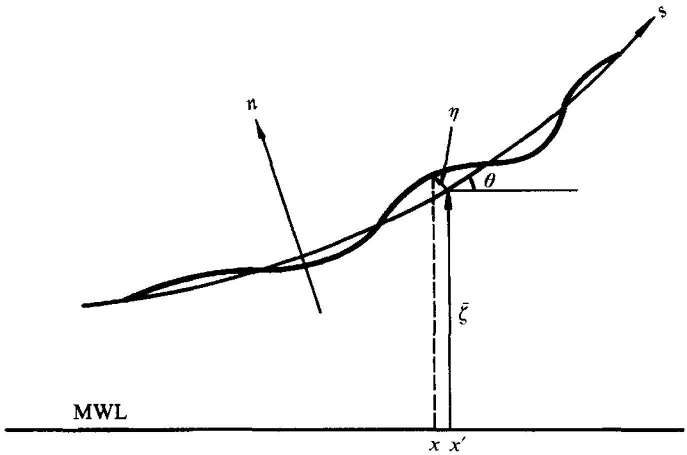

FIGURE 3. Definition sketch.

neglecting the nonlinear short-wave terms. In (2.8), $W|_{n=0}$ was also neglected. It is zero in the curvilinear coordinates in the absence of the short-wave disturbance, but will be of second order in the slope of the short wave in the present case, as required to balance the mean value of the nonlinear terms of short waves in (2.7). This term corresponds to a mean mass flux across the undisturbed surface of the long wave suggested by Hasselmann (1971). On the other hand, the term $\eta\,\partial W/\partial n|_{n=0}$ is $O((a/l)(Al/L^2))$ (a and l represent respectively the typical amplitude and wavelength of the short wave), which is intimately related to the modulation of the short waves by the longer one, and is not negligible in the following discussion. It is also interesting to notice that in this approximation we do not require A/L to be small, because no linear term of the short wave with higher order in A/L has been neglected. In other words, the underlying long waves can be of finite amplitude.

From the continuity equation in curvilinear coordinates, we also have

$$\frac{\partial W}{\partial n} = -\frac{\partial U}{\partial s} \quad \text{at} \quad n = 0,$$

because $h_s|_{n=0} = 1$. Hence an approximate kinematical boundary condition for the linear short-wave field superimposed upon a steady long, finite-amplitude wave can finally be written as

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial s} - \frac{\partial \phi}{\partial n} + \eta\frac{\partial U}{\partial s} = 0 \quad \text{at} \quad n = 0. \tag{2.9}$$

An analogous approximation for the dynamical boundary condition can also be found. At first, from the condition of a constant atmospheric pressure and Bernoulli's equation, we have exactly

$$\frac{\partial \phi}{\partial t} + g(\bar{\zeta} + \eta\cos\theta) - \gamma F + \frac{1}{2}\left[\left(U + \frac{1}{h_s}\frac{\partial \phi}{\partial s}\right)^2 + \left(W + \frac{\partial \phi}{\partial n}\right)^2\right] = \text{const.} \quad \text{at} \quad n = \eta, \tag{2.10}$$

where $\bar{\zeta}$ and θ are respectively the surface elevation and slope induced by the long wave (figure 3), and F is the curvature of the instantaneous surface. The surface-tension term after expansion can be reduced to

$$-\gamma F = -\gamma \frac{\partial^2 \bar{\zeta}/\partial x'^2}{[1 + (\partial \bar{\zeta}/\partial x')^2]^{\frac{3}{2}}} - \gamma \frac{\partial^2 \eta}{\partial s^2} + O\left(\frac{a^2}{l^2}\frac{A}{L}, \frac{a^2}{l^2}\frac{Al}{L^2}\right), \tag{2.11}$$

where x' measures the distance in the horizontal direction (see figure 3 for the difference between x and x'). Notice that in (2.11) the terms

$$\gamma\left(\frac{\partial\bar{\zeta}}{\partial x}\right)^2\frac{\partial^2\eta}{\partial s^2},\quad \gamma\frac{\partial\bar{\zeta}}{\partial x}\frac{\partial^2\bar{\zeta}}{\partial x^2}\frac{\partial\eta}{\partial s}\quad\text{and}\quad \gamma\left(\frac{\partial\bar{\zeta}}{\partial x}\right)^3\frac{\partial^2\bar{\zeta}}{\partial x^2}\frac{\partial\eta}{\partial s}$$

are cancelled out after the variations of $\bar{\zeta}$ and η in space are all expressed by their derivatives with respect to x' and s respectively, otherwise the above approximation will be insufficient for finite-amplitude long waves. We also recognized that it is $\partial/\partial x'$ instead of $\partial/\partial x$ which represents the correct operator for calculating the rate of change of $\bar{\zeta}$ in space, because the changes of x and the position where $\bar{\zeta}$ is measured do not exactly equal each other. Now substituting (2.1) and (2.11) in (2.10), expanding the resulting equation, and neglecting the higher-order terms in a/l, we obtain

$$\left[g\bar{\zeta}-\gamma\frac{\partial^2\bar{\zeta}/\partial x'^2}{\{1+(\partial\bar{\zeta}/\partial x')^2\}^{\frac{3}{2}}}+\tfrac{1}{2}U^2\right]+\left[\frac{\partial\phi}{\partial t}+g'\eta-\gamma\frac{\partial^2\eta}{\partial s^2}+U\frac{\partial\phi}{\partial s}\right]=\text{const.}\quad\text{at}\quad n=0, \tag{2.12}$$

where
$$g' = g\cos\theta+U^2/R \tag{2.13}$$

is the effective gravitational acceleration suggested by Phillips (1981). In (2.13), the term $U\partial U/\partial n$ is replaced by U^2/R because $\partial U/\partial n = U/R$ in an irrotational flow.

In (2.12), without the short-wave disturbance the sum of the terms in the first square brackets is constant on account of a constant atmospheric pressure, but in the present case it may vary to balance a non-zero mean value of the nonlinear terms of the short wave, which implies a mean pressure acting on the boundary of the long-wave field and therefore a modification to the latter. However, we here neglect all the nonlinear effects of the short wave, therefore from (2.12) we have for the short-wave motion

$$\frac{\partial\phi}{\partial t}+g'\eta-\gamma\frac{\partial^2\eta}{\partial s^2}+U\frac{\partial\phi}{\partial s}=0\quad\text{at}\quad n=0. \tag{2.14}$$

Equations (2.9) and (2.14) are the two approximate boundary conditions for linear short waves propagating on a steady finite-amplitude long wave, or on a steady larger-scale irrotational current. However, no two-scale approximation has so far been made. We shall next combine the two equations (2.9) and (2.14) into one for the surface displacement η. To accomplish this, it is sufficient to take advantage of the fact that the apparent frequencies n_0 of the incident and reflected waves are the same. Therefore the time-dependent parts of the solutions for η and ϕ can be separated and represented by the factor $\exp(-in_0 t)$. Consequently the two boundary conditions (2.9) and (2.14) can be rewritten as

$$-in_0\eta+U\frac{\partial\eta}{\partial s}-\frac{\partial\phi}{\partial n}+\eta\frac{\partial U}{\partial s}=0, \tag{2.15}$$

and
$$-in_0\phi+g'\eta-\gamma\frac{\partial^2\eta}{\partial s^2}+U\frac{\partial\phi}{\partial s}=0. \tag{2.16}$$

Differentiation of (2.16) produces

$$-in_0\frac{\partial\phi}{\partial s}+g'\frac{\partial\eta}{\partial s}+\eta\frac{\partial g'}{\partial s}-\gamma\frac{\partial^3\eta}{\partial s^3}+\frac{\partial}{\partial s}\left(U\frac{\partial\phi}{\partial s}\right)=0. \tag{2.17}$$

Scale ratio	Magnitude	Physical significance
a/l	$\ll 1$	Wave slope of short wave
A/L	≈ 1	Wave slope of long wave
l/L	$\ll 1$	Ratio of two lengthscales
Al/L^2	$\ll 1$	Rate of modulation

TABLE 1. Magnitudes of the scale ratios

In (2.17) a new term $\eta\, \partial g'/\partial s$ is introduced and it follows from (2.13) and figure 3 that

$$\eta \frac{\partial g'}{\partial s} \approx \eta g \frac{\partial}{\partial s}(\cos\theta) \approx -\eta g \frac{\partial \zeta}{\partial s}\frac{\partial^2 \zeta}{\partial s^2},$$

which is of order $(a/l)(Al/L^2)(A/L)$ and should not be neglected when the underlying long wave is of finite amplitude. Next, from (2.6) and (2.15),

$$\frac{\partial \phi}{\partial s} = \mathrm{i}\frac{\partial \phi}{\partial n} = \mathrm{i}\left(-\mathrm{i}n_0\,\eta + U\frac{\partial \eta}{\partial s} + \eta\frac{\partial U}{\partial s}\right).$$

Substitution in (2.17) for $\partial\phi/\partial s$ and expanding yields finally

$$\frac{\partial^3 \eta}{\partial s^3} - \mathrm{i}\frac{U^2}{\gamma}\frac{\partial^2 \eta}{\partial s^2} - \frac{1}{\gamma}\left(2n_0\,U + g' + 3\mathrm{i}U\frac{\partial U}{\partial s}\right)\frac{\partial \eta}{\partial s} - \frac{1}{\gamma}\left(-\mathrm{i}n_0^2 + 2n_0\frac{\partial U}{\partial s} + \frac{\partial g'}{\partial s}\right)\eta = 0 \quad (2.18)$$

evaluated at $n = 0$. In obtaining (2.18), the terms $\mathrm{i}\eta U\,\partial^2 U/\partial s^2$ and $\mathrm{i}\eta(\partial U/\partial s)^2$ with orders $(a/l)(Al/L^2)(l/L)$ and $(a/l)(Al/L^2)^2$ respectively are neglected, because we will later calculate the uniformly valid asymptotic solution only to $O((a/l)(Al/L^2))$. Thus (2.18) represents a two-scale approximation for $l/L \ll 1$. The assumed magnitudes of all the scale ratios mentioned above are summarized in table 1 for reference.

Equation (2.18) with $\gamma \neq 0$ is a third-order ordinary differential equation for η only, in which the influence of the long wave on the short wave is implied by the slowly varying coefficients involving U and g' as well as their derivatives. Such an equation in general cannot be solved exactly, and no systematic method exists to obtain its asymptotic solution describing the blockage process. However, the suggestion of decomposing a higher-order equation and from it to extract a lower-order one in an asymptotic analysis has been made by Turrittin (1952). Since in the case of blockage, only a pair of incident and reflected waves is involved, we shall in the next section extract from (2.18) a second-order differential equation describing this pair of waves. One may therefore expect that the resulting equation can fully describe the blockage phenomenon as well as be virtually solvable.

In the coming discussion, we replace the symbols (s, n) by (x, z) to make the notation more conventional. Accordingly, (2.18) can be written as

$$\frac{\partial^3 \eta}{\partial x^3} - \mathrm{i}\frac{U^2}{\gamma}\frac{\partial^2 \eta}{\partial x^2} - \frac{1}{\gamma}\left(2n_0\,U + g' + 3\mathrm{i}U\frac{\partial U}{\partial x}\right)\frac{\partial \eta}{\partial x} - \frac{1}{\gamma}\left(-\mathrm{i}n_0^2 + 2n_0\frac{\partial U}{\partial x} + \frac{\partial g'}{\partial x}\right)\eta = 0 \quad \text{at} \quad z = 0. \quad (2.19)$$

3. The equation coupling the incident and reflected waves

The procedure of decomposition starts by rewriting (2.19) as an equivalent first-order system of differential equations. This is done by introducing the higher derivatives of the dependent variable η as additional unknown functions. Therefore in matrix notation we define a column matrix

$$\bar{\boldsymbol{\eta}} = \begin{bmatrix} \eta \\ \eta' \\ \eta'' \end{bmatrix} \tag{3.1}$$

(in the following discussion we shall use the shorthand notation $\eta' = \partial\eta/\partial x$, $\eta'' = \partial^2\eta/\partial x^2$, $k' = \partial k/\partial x$, etc. at certain places), then equation (2.19) takes on the concise form

$$\bar{\boldsymbol{\eta}}' = (\boldsymbol{B}_0 + \boldsymbol{B}_1)\,\bar{\boldsymbol{\eta}}, \tag{3.2}$$

with its coefficient matrices

$$\boldsymbol{B}_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ d_1 & d_2 & d_3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_4 & d_5 & 0 \end{bmatrix}, \tag{3.3}$$

where

$$\left.\begin{aligned} d_1 &= -\mathrm{i}\frac{n_0^2}{\gamma}, \quad d_2 = \frac{1}{\gamma}(2n_0 U + g'), \quad d_3 = \mathrm{i}\frac{U^2}{\gamma}, \\ d_4 &= \frac{2n_0}{\gamma}\frac{\partial U}{\partial x} + \frac{1}{\gamma}\frac{\partial g'}{\partial x}, \quad d_5 = 3\mathrm{i}\frac{U}{\gamma}\frac{\partial U}{\partial x}. \end{aligned}\right\} \tag{3.4}$$

Notice that in (3.2) the second square matrix $\boldsymbol{B}_1$ is deliberately separated from $\boldsymbol{B}_0$ and involves only smaller quantities; this treatment will later be found very helpful.

Now if both matrices $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are block-diagonalized into two diagonal blocks of orders 1 and 2, then the original system (3.2) can be split into two uncoupled systems of orders 1 and 2. This might be achieved by a transformation of the dependent variable $\bar{\boldsymbol{\eta}}$, such as

$$\bar{\boldsymbol{\eta}} = \boldsymbol{S}\bar{\boldsymbol{y}}, \tag{3.5}$$

where $\boldsymbol{S}$ is a 3×3 matrix and remains to be defined. In order to determine an appropriate $\boldsymbol{S}$, we first substitute (3.5) into (3.2), which becomes

$$\boldsymbol{S}\bar{\boldsymbol{y}}' + \boldsymbol{S}'\bar{\boldsymbol{y}} = (\boldsymbol{B}_0 + \boldsymbol{B}_1)\boldsymbol{S}\bar{\boldsymbol{y}}.$$

Multiplying both sides by the inverse matrix $\boldsymbol{S}^{-1}$, we have a new differential equation

$$\bar{\boldsymbol{y}}' = (\boldsymbol{C}_0 + \boldsymbol{C}_1)\,\bar{\boldsymbol{y}}, \tag{3.6}$$

with

$$\boldsymbol{C}_0 = \boldsymbol{S}^{-1}\boldsymbol{B}_0\boldsymbol{S} \tag{3.7}$$

and

$$\boldsymbol{C}_1 = \boldsymbol{S}^{-1}\boldsymbol{B}_1\boldsymbol{S} - \boldsymbol{S}^{-1}\boldsymbol{S}'. \tag{3.8}$$

in (3.6), the new second matrix $\boldsymbol{C}_1$ also involves only smaller quantities, for the differentiation $\boldsymbol{S}'$ increases the order by one level as $\boldsymbol{S}$ is presumably composed of slowly varying coefficients. Expression (3.7) represents a similarity transformation of the square matrix $\boldsymbol{B}_0$. It turns out that if each column of the transformation matrix $\boldsymbol{S}$ consists of the components of each different eigenvector of $\boldsymbol{B}_0$, then the new matrix $\boldsymbol{C}_0$ in (3.7) will be diagonal as required.

From (3.3) the eigenvalues λ_1, λ_2 and λ_3 of $\boldsymbol{B}_0$ satisfy

$$0 = \det(\boldsymbol{B}_0 - \lambda \boldsymbol{I}) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ d_1 & d_2 & d_3-\lambda \end{vmatrix}$$

$$= -\lambda^3 + d_3\lambda^2 + d_2\lambda + d_1, \tag{3.9}$$

where $\boldsymbol{I}$ is the unit matrix. Equation (3.9) is a cubic polynomial; its three solutions have the following relations:

$$\left.\begin{aligned} \lambda_1+\lambda_2+\lambda_3 &= d_3, \\ \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 &= -d_2 \\ \lambda_1\lambda_2\lambda_3 &= d_1. \end{aligned}\right\} \tag{3.10}$$

It is interesting to notice that there are closed relations between λ_j and the three wavenumbers k_j satisfying the dispersion relation

$$n_0 = (g'k+\gamma k^3)^{\frac{1}{2}} + Uk.$$

An expansion of the above equation yields

$$k^3 - \frac{U^2}{\gamma}k^2 + \frac{g'+2n_0 U}{\gamma}k - \frac{n_0^2}{\gamma} = 0. \tag{3.11}$$

Comparison with (3.9) and (3.4) immediately leads to

$$\lambda_j = \mathrm{i}k_j \quad \text{for} \quad j = 1,2,3. \tag{3.12}$$

(The third solution k_3 corresponds to the intersection of the curve $\sigma = -(g'k+\gamma k^3)^{\frac{1}{2}}$ with the straight line (1.3) in figure 1, and therefore represents a wave intrinsically propagating downstream, though with the same apparent frequency n_0.) The closed relations between λ_j and k_j are certainly not accidental, which is encouraging for the current approach.

By using (3.10) and the ordinary procedure for calculating eigenvectors, we may obtain the three eigenvectors of $\boldsymbol{B}_0$

$$e_j = \begin{bmatrix} 1 \\ \lambda_j \\ \lambda_j^2 \end{bmatrix} \quad \text{for} \quad j = 1,2,3.$$

Consequently the appropriate transformation matrix $\boldsymbol{S}$ is

$$\boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}. \tag{3.13}$$

From (3.13) we may calculate the inverse matrix $\boldsymbol{S}^{-1}$ as well as $\det \boldsymbol{S}$ and, as mentioned previously, upon choice of the specific form (3.13) of the transformation matrix, the new matrix $\boldsymbol{C}_0$ in (3.7) becomes diagonal. Actually, the diagonal elements are the eigenvalues λ_j which are invariants of the matrix. Therefore we have

$$\boldsymbol{C}_0 = \boldsymbol{S}^{-1}\boldsymbol{B}_0\boldsymbol{S} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{3.14}$$

So far we have only succeeded in block-diagonalizing the first square matrix of (3.6). On substituting $\boldsymbol{S}$, $\boldsymbol{S}^{-1}$ and the derivative $\boldsymbol{S}'$ into (3.8), the second coefficient matrix is obtained:

$$\boldsymbol{C}_1 = \boldsymbol{S}^{-1}\boldsymbol{B}_1\boldsymbol{S}-\boldsymbol{S}^{-1}\boldsymbol{S}' = \frac{1}{\det \boldsymbol{S}} \times$$
$$\begin{bmatrix} (\lambda_3-\lambda_2)[d_4+d_5\lambda_1 & (\lambda_3-\lambda_2)[d_4+d_5\lambda_2 & (\lambda_3-\lambda_2)[d_4+d_5\lambda_3 \\ \quad -\lambda_1'(2\lambda_1-\lambda_2-\lambda_3)] & \quad +\lambda_2'(\lambda_3-\lambda_2)] & \quad -\lambda_3'(\lambda_3-\lambda_2)] \\ (\lambda_1-\lambda_3)[d_4+d_5\lambda_1 & (\lambda_1-\lambda_3)[d_4+d_5\lambda_2 & (\lambda_1-\lambda_3)[d_4+d_5\lambda_3 \\ \quad -\lambda_1'(\lambda_1-\lambda_3)] & \quad -\lambda_2'(2\lambda_2-\lambda_1-\lambda_3)] & \quad +\lambda_3'(\lambda_1-\lambda_3)] \\ (\lambda_2-\lambda_1)[d_4+d_5\lambda_1 & (\lambda_2-\lambda_1)[d_4+d_5\lambda_2 & (\lambda_2-\lambda_1)[d_4+d_5\lambda_3 \\ \quad +\lambda_1'(\lambda_2-\lambda_1)] & \quad -\lambda_2'(\lambda_2-\lambda_1)] & \quad -\lambda_3'(2\lambda_3-\lambda_1-\lambda_2)] \end{bmatrix}.$$

Obviously $\boldsymbol{C}_1$ is not block diagonal, therefore another transformation of the dependent variable

$$\bar{y} = \boldsymbol{T}\bar{\xi} \tag{3.15}$$

is needed. We now assume the transformation matrix

$$\boldsymbol{T} = \boldsymbol{T}_0+\boldsymbol{T}_1+\ldots, \tag{3.16}$$

which represents an asymptotic expansion such that $\boldsymbol{T}_j$ are terms of successively smaller order in the relevant small parameters. However, for the present purposes only the first two terms $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$ are required to achieve the same level of approximation as we have in §2. Substituting (3.15) and (3.16) into (3.6) we have

$$(\boldsymbol{T}_0+\boldsymbol{T}_1+\ldots)\bar{\xi}' = [(\boldsymbol{C}_0+\boldsymbol{C}_1)(\boldsymbol{T}_0+\boldsymbol{T}_1+\ldots)-(\boldsymbol{T}_0'+\boldsymbol{T}_1'+\ldots)]\bar{\xi}. \tag{3.17}$$

If the resulting differential matrix equation is represented by

$$\bar{\xi}' = (\boldsymbol{D}_0+\boldsymbol{D}_1+\ldots)\bar{\xi}, \tag{3.18}$$

then expanding and equalizing the terms of equal order to zero lead to the recursion formulae

$$\boldsymbol{T}_0\boldsymbol{D}_0 = \boldsymbol{C}_0\boldsymbol{T}_0, \tag{3.19}$$

$$\boldsymbol{T}_0\boldsymbol{D}_1+\boldsymbol{T}_1\boldsymbol{D}_0 = \boldsymbol{C}_0\boldsymbol{T}_1+\boldsymbol{C}_1\boldsymbol{T}_0-\boldsymbol{T}_0'. \tag{3.20}$$

In obtaining (3.20), recall that each differentiation of the coefficients increases the order by one.

From (3.19), if we set $\boldsymbol{T}_0 = \boldsymbol{I}$, the unit matrix, then

$$\boldsymbol{D}_0 = \boldsymbol{C}_0 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{3.21}$$

In other words, the first diagonal matrix remains unchanged in the latter transformation. In this circumstance, equation (3.20) can also be simplified to

$$\boldsymbol{D}_1 = \boldsymbol{C}_0\boldsymbol{T}_1-\boldsymbol{T}_1\boldsymbol{C}_0+\boldsymbol{C}_1. \tag{3.22}$$

Now the aim is to determine $\boldsymbol{T}_1$ such that $\boldsymbol{D}_1$ can become block diagonal. The technique for accomplishing this was described by Wasow (1985). By using this technique, we finally obtain

$$\boldsymbol{T}_1 = \frac{1}{\det \boldsymbol{S}} \begin{bmatrix} 0 & 0 & \dfrac{\lambda_3-\lambda_2}{\lambda_3-\lambda_1}[d_4+d_5\lambda_3 - \lambda_3'(\lambda_3-\lambda_2)] \\ 0 & 0 & \dfrac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}[d_4+d_5\lambda_3 + \lambda_3'(\lambda_1-\lambda_3)] \\ \dfrac{\lambda_1-\lambda_2}{\lambda_3-\lambda_1}[d_4+d_5\lambda_1 - \lambda_1'(\lambda_1-\lambda_2)] & \dfrac{\lambda_1-\lambda_2}{\lambda_3-\lambda_2}[d_4+d_5\lambda_2 + \lambda_2'(\lambda_1-\lambda_2)] & 0 \end{bmatrix}. \tag{3.23}$$

The fitness of (3.23) can easily be verified by substituting it, as well as $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$, into (3.22), giving

$$\boldsymbol{D}_1 = \frac{1}{\det \boldsymbol{S}} \begin{bmatrix} (\lambda_3-\lambda_2)[d_4+d_5\lambda_1 - \lambda_1'(2\lambda_1-\lambda_2-\lambda_3)] & (\lambda_3-\lambda_2)[d_4+d_5\lambda_2 + \lambda_2'(\lambda_3-\lambda_2)] & 0 \\ (\lambda_1-\lambda_3)[d_4+d_5\lambda_1 - \lambda_1'(\lambda_1-\lambda_3)] & (\lambda_1-\lambda_3)[d_4+d_5\lambda_2 - \lambda_2'(2\lambda_2-\lambda_1-\lambda_3)] & 0 \\ 0 & 0 & (\lambda_2-\lambda_1)[d_4+d_5\lambda_3 \; \lambda_3'(2\lambda_3-\lambda_1-\lambda_2)] \end{bmatrix}, \tag{3.24}$$

which is indeed block diagonal.

Since both matrices $\boldsymbol{D}_0$ and $\boldsymbol{D}_1$ are now transformed into two diagonal blocks of orders 2 and 1, the system (3.18) with the neglect of the higher-order coefficient matrix can be formally split into two uncoupled systems of orders 2 and 1. We may first set the column matrix

$$\bar{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix},$$

then the substitution of (3.21) and (3.24) into (3.18) produces two systems, one of which is the first-order equation for ξ_3:

$$\xi_3' = \left\{\lambda_3 + \frac{(\lambda_2-\lambda_1)}{\det \boldsymbol{S}}[d_4+d_5\lambda_3 - \lambda_3'(2\lambda_3-\lambda_1-\lambda_2)]\right\}\xi_3,$$

or equivalently
$$\left[\frac{\partial}{\partial x} - (\lambda_3+E_3)\right]\xi_3 = 0, \tag{3.25}$$

with
$$E_3 = \frac{(\lambda_2-\lambda_1)}{\det \boldsymbol{S}}[d_4+d_5\lambda_3 - \lambda_3'(2\lambda_3-\lambda_1-\lambda_2)].$$

Next, an inverse transformation from $\bar{\xi}$ to $\bar{\eta}$ will be performed. From (3.5), (3.15) and (3.16) we have the following relation:

$$\bar{\boldsymbol{\eta}} = \boldsymbol{S}(\boldsymbol{T}_0 + \boldsymbol{T}_1)\bar{\xi}.$$

Thus an inverse transformation yields

$$\bar{\xi} = (\boldsymbol{T}_0+\boldsymbol{T}_1)^{-1}\boldsymbol{S}^{-1}\bar{\boldsymbol{\eta}}.$$

By a straightforward calculation, we obtain

$$\xi_3 = \frac{1}{(\det\boldsymbol{S})^2}(E_2\eta''+E_1\eta'+E_0\eta), \tag{3.26}$$

with

$$E_0 = \det\boldsymbol{S}(\lambda_1\lambda_2^2-\lambda_2\lambda_1^2)+\frac{\lambda_2\lambda_3^2-\lambda_3\lambda_2^2}{\lambda_3-\lambda_1}\pi_1+\frac{\lambda_3\lambda_1^2-\lambda_1\lambda_3^2}{\lambda_3-\lambda_2}\pi_2,$$

$$E_1 = \det\boldsymbol{S}(\lambda_1^2-\lambda_2^2)+\frac{\lambda_2^2-\lambda_3^2}{\lambda_3-\lambda_1}\pi_1+\frac{\lambda_3^2-\lambda_1^2}{\lambda_3-\lambda_2}\pi_2,$$

$$E_2 = \det\boldsymbol{S}(\lambda_2-\lambda_1)+\frac{\lambda_3-\lambda_2}{\lambda_3-\lambda_1}\pi_1+\frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}\pi_2,$$

where

$$\pi_1 = (d_4+d_5\lambda_1)(\lambda_2-\lambda_1)+(\lambda_1-\lambda_2)^2\lambda_1',$$

$$\pi_2 = (d_4+d_5\lambda_2)(\lambda_2-\lambda_1)-(\lambda_1-\lambda_2)^2\lambda_2'.$$

Substitution into (3.25) yields

$$\left[\frac{\partial}{\partial x}-(\lambda_3+E_3)\right]\frac{1}{(\det\boldsymbol{S})^2}(E_2\eta''+E_1\eta'+E_0\eta) = 0.$$

This equation is equivalent to the original third-order differential equation (2.19) if we neglect those higher-order terms in the asymptotic expansion. However, the present equation has been factorized; and in order to study the reflection phenomenon we may seek the solutions satisfying the second-order differential equation

$$E_2\eta''+E_1\eta'+E_0\eta = 0.$$

Division by E_2, and the neglect of higher-order terms in l/L, leads to

$$\eta''+[-\mathrm{i}(k_1+k_2)+Q]\eta'+[-k_1k_2+P]\eta = 0, \tag{3.27}$$

where

$$Q = \frac{-1}{(k_3-k_1)(k_3-k_2)}[d_4+\mathrm{i}k_3d_5+(k_3-k_2)k_1'+(k_3-k_1)k_2'], \tag{3.28}$$

$$P = \frac{-1}{(k_3-k_1)(k_3-k_2)}[\mathrm{i}(k_3-k_2-k_1)d_4+k_1k_2d_5-\mathrm{i}k_2(k_3-k_2)k_1'-\mathrm{i}k_1(k_3-k_1)k_2']. \tag{3.29}$$

In (3.27)–(3.29), λ_j have been replaced by k_j according to (3.12).

Equation (3.27), especially its dominant terms, indicates that its two independent solutions correspond to the k_1 and k_2 components. Furthermore, if k_1 and k_2 represent a pair of the anticipated incident and reflected waves (figure 1), then it can be proved that the singular properties of k_1 and k_2 at the blockage point are completely offset in (3.27) so that it is essentially regular at the blockage point. Equation (3.27) as a rigorous two-scale asymptotic approximation is therefore uniformly valid. We also note that further decomposition of (3.27) could be continued, but will result in two first-order differential equations, both singular at the blockage point; and this is pointless for the present purpose, since the combination of the two solutions is our primary interest.

4. Solutions of a reflected linear wave

There are many works which provide uniformly valid asymptotic solutions for second-order ordinary differential equations with slowly varying coefficients (see, for example, Olver 1974). However, since the treatment for each equation is slightly different and since for the present purposes only the leading terms in the asymptotic expansion of the solution are required, we shall here develop a quick and clear derivation suggested by Smith's (1975) results.

First, we eliminate the first derivative term from (3.27) by a change of the dependent variable η, such that

$$\eta = v(x)\,\mathrm{e}^{-\mathrm{i}n_0 t}\exp\left\{-\frac{1}{2}\int[-\mathrm{i}(k_1+k_2)+Q]\,\mathrm{d}x\right\}, \tag{4.1}$$

where $v(x)$ is the new dependent variable. Now, substituting (4.1) into (3.27), neglecting higher-order terms of approximation involving k_1'', k_2'', d_4' or d_5', and also crossing out the common factor, we obtain

$$v''+v(H+G) = 0, \tag{4.2}$$

where
$$H = \tfrac{1}{4}(k_1-k_2)^2, \tag{4.3}$$

$$G = \frac{-1}{(k_3-k_1)(k_3-k_2)}[\mathrm{i}(k_3-\tfrac{1}{2}k_2-\tfrac{1}{2}k_1)\,d_4-(\tfrac{1}{2}k_1k_3+\tfrac{1}{2}k_2k_3-k_1k_2)\,d_5 \\ -\mathrm{i}(k_3-k_2)(\tfrac{1}{2}k_3+\tfrac{1}{2}k_2-k_1)\,k_1'-\mathrm{i}(k_3-k_1)(\tfrac{1}{2}k_3+\tfrac{1}{2}k_1-k_2)\,k_2']. \tag{4.4}$$

Equation (4.2) can be proved to be still regular at the blockage point, and from Smith's (1975) results, we may expect that

$$v \approx A_0\,\mathrm{Ai}\,(-r)-C_0\,\mathrm{Ai}'\,(-r), \tag{4.5}$$

where

$$\left.\begin{aligned} \tfrac{2}{3}r^{\frac{3}{2}} &= \int_0^x H^{\frac{1}{2}}\,\mathrm{d}x, \\ A_0 &= \left(\frac{r}{H}\right)^{\frac{1}{4}}\cos\left(\int_0^x \tfrac{1}{2}G/H^{\frac{1}{2}}\,\mathrm{d}x\right), \\ C_0 &= r^{-\frac{1}{4}}H^{-\frac{1}{4}}\sin\left(\int_0^x \tfrac{1}{2}G/H^{\frac{1}{2}}\,\mathrm{d}x\right). \end{aligned}\right\} \tag{4.6}$$

For the sake of definiteness, we have taken $x = 0$ to be the blockage point and assumed that $H > 0$ for $x > 0$. In (4.5), Ai and Ai′ represent the Airy function and its derivative; the former satisfies the equation

$$\frac{\mathrm{d}^2\,\mathrm{Ai}\,(-r)}{\mathrm{d}r^2}+r\,\mathrm{Ai}\,(-r) = 0. \tag{4.7}$$

The fitness of (4.5) and (4.6) can easily be verified by substituting them into (4.2), which results in

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+v(H+G) = \frac{\mathrm{d}^2A_0}{\mathrm{d}x^2}\mathrm{Ai}\,(-r)-\frac{\mathrm{d}^2C_0}{\mathrm{d}x^2}\mathrm{Ai}'\,(-r) \tag{4.8}$$

as $\mathrm{Ai}''(-r)$ is eliminated in favour of $-r\,\mathrm{Ai}\,(-r)$. The differences between (4.2) and (4.8) are insignificant, because the coefficients A_0 and C_0 have been separated from the rapidly varying parts of the solution – the new variable r also fulfills the condition of slow modulation (but would not do so if r contained a very high power of H or some exponential functions like $\exp\left(\int_0^x H^{\frac{1}{2}}\,\mathrm{d}x\right)$). This and the fact that both A_0 and C_0 are regular at the blockage point ensure that the second derivatives of A_0 and C_0 are negligible everywhere when compared with H and G. We also note that though H vanishes at the blockage point, G will in general not vanish at the same point. Thus the expressions (4.5) and (4.6) together with (4.1) represent a uniformly valid asymptotic solution of η.

At points away from the blockage point, $\mathrm{Ai}(-r)$ and $\mathrm{Ai}'(-r)$ can be replaced by their asymptotic expansions, which for r large and positive are

$$\mathrm{Ai}(-r) = \pi^{-\frac{1}{2}} r^{-\frac{1}{4}} \sin\left(\tfrac{2}{3} r^{\frac{3}{2}} + \tfrac{1}{4}\pi\right),$$

and

$$\mathrm{Ai}'(-r) = -\pi^{-\frac{1}{2}} r^{\frac{1}{4}} \cos\left(\tfrac{2}{3} r^{\frac{3}{2}} + \tfrac{1}{4}\pi\right).$$

(For r large and negative, both $\mathrm{Ai}(-r)$ and $\mathrm{Ai}'(-r)$ have a decreasing exponential behaviour, therefore (4.5) indeed represents the acceptable solution.) Thus from (4.5), (4.6) and (4.1) we have

$$\eta \approx H^{-\frac{1}{4}} \exp\left[\int_0^x \tfrac{1}{2}(-Q - \mathrm{i}G/H^{\frac{1}{2}})\,\mathrm{d}x\right] \exp \mathrm{i}\left[\int_0^x k_1\,\mathrm{d}x - n_0 t + \tfrac{1}{4}\pi\right]$$
$$+ H^{-\frac{1}{4}} \exp\left[\int_0^x \tfrac{1}{2}(-Q + \mathrm{i}G/H^{\frac{1}{2}})\,\mathrm{d}x\right] \exp \mathrm{i}\left[\int_0^x k_2\,\mathrm{d}x - n_0 t - \tfrac{1}{4}\pi\right] \quad (4.9)$$

for $x \gg 0$. The solution (4.9) actually represents the WKBJ solution, which consists of an incident wave k_1 and a reflected wave k_2 (here we have chosen $k_1 < k_2$). Unlike (4.5), this solution obviously fails at the blockage point where $H = 0$, but it has provided the relative magnitudes and phases of the incident and reflected waves (an irrelevant constant common factor was neglected from (4.9)). Consequently, we have the local amplitudes

$$a = \begin{cases} H^{-\frac{1}{4}} \exp\left[\int_0^x \tfrac{1}{2}(-Q - \mathrm{i}G/H^{\frac{1}{2}})\,\mathrm{d}x\right] & (k_1 \text{ component}); \\ H^{-\frac{1}{4}} \exp\left[\int_0^x \tfrac{1}{2}(-Q + \mathrm{i}G/H^{\frac{1}{2}})\,\mathrm{d}x\right] & (k_2 \text{ component}). \end{cases} \quad (4.10)$$

Notice that G is pure imaginary while H and Q are real.

For individual incident and reflected waves, it can be shown that the variation of a described in (4.10) satisfies the action conservation principle (which was first established by Bretherton & Garrett 1968 for a general class of situations, but the latest proof of it for the propagation of linearized short waves on a longer wave of finite amplitude was given by Henyey *et al.* 1988). Since it involves a great deal of algebraic manipulation, a proof is described in Appendix A. From (A 10) and (A 11) we have

$$\frac{\partial}{\partial x}\left[(U + C_{g1})\frac{E_1}{\sigma_1}\right] = 0, \quad \frac{\partial}{\partial x}\left[(U + C_{g2})\frac{E_2}{\sigma_2}\right] = 0, \quad (4.11)$$

which in the present case with a steady larger-scale flow indeed represent the action conservation equation. Thus for each component, the present WKBJ solution is consistent with the earlier ray solution. However, it should be emphasized that the above solution and the proof of (4.11) have incorporated the influence of a variable effective gravity g', and provide deeper insight into the nature of wave modulation. Expansion of (4.11) yields

$$\frac{1}{\sigma}\frac{\partial}{\partial x}[(U+C_g)E]+(U+C_g)E\frac{\partial}{\partial x}\left(\frac{1}{\sigma}\right)=0. \tag{4.12}$$

(The subscripts 1 and 2 have been dropped for a general discussion.) Since $\sigma = (g'k+\gamma k^3)^{\frac{1}{2}}$ and $C_g \equiv \partial\sigma/\partial k$, we have further

$$\frac{\partial}{\partial x}\left(\frac{1}{\sigma}\right) = -\frac{1}{\sigma^2}\left(\frac{\partial\sigma}{\partial k}\frac{\partial k}{\partial x}+\frac{\partial\sigma}{\partial g'}\frac{\partial g'}{\partial x}\right)$$

$$= -\frac{1}{\sigma^2}\left(C_g\frac{\partial k}{\partial x}+\frac{1}{2}\frac{k}{\sigma}\frac{\partial g'}{\partial x}\right).$$

As explained earlier, the term $\partial g'/\partial x$ cannot be neglected if the slope of the undisturbed surface changes significantly. In fact, in Appendix A, the action-conservation equation (4.11) is shown to be correct even when $\partial g'/\partial x$ is included. Equation (4.12) can be written as

$$\frac{\partial}{\partial x}[(U+C_g)E]-\frac{1}{\sigma}(U+C_g)E\left[C_g\frac{\partial k}{\partial x}+\frac{1}{2}\frac{k}{\sigma}\frac{\partial g'}{\partial x}\right]=0. \tag{4.13}$$

From (A 9), the differentiation $\partial k/\partial x$ can be further expressed as

$$\frac{\partial k}{\partial x} = -\frac{k}{U+C_g}\frac{\partial U}{\partial x}-\frac{1}{2}\frac{1}{U+C_g}\frac{k}{\sigma}\frac{\partial g'}{\partial x},$$

and substitution in (4.13) results in

$$\frac{\partial}{\partial x}[(U+C_g)E]+EC_g\frac{k}{\sigma}\frac{\partial U}{\partial x}-\tfrac{1}{2}UE\frac{k}{\sigma^2}\frac{\partial g'}{\partial x}=0. \tag{4.14}$$

In (4.14), the first and second terms are familiar to us and represent respectively the energy flux and the work done by the radiation stress, but the third term is extra. It indicates that if the undisturbed surface has significantly varied slope, an additional energy interaction between short waves and long waves will take place; although the wave action is conserved in all cases.

Next, one may ask whether or not the action is also conserved on reflection of gravity–capillary waves from the blockage point. According to Smith (1975), the answer for gravity waves is positive, but this conclusion was drawn from a requirement of cancellation of the singularities in the approximate solution. In the present approach, this important feature can be demonstrated explicitly. At first, from (A 1) and (4.10), the action fluxes of the incident and reflected waves are

$$(U+C_g)\frac{E}{\sigma} = \begin{cases} (U+C_{g1})\dfrac{\sigma_1}{k_1}H^{-\frac{1}{2}}\exp\left[\displaystyle\int_0^x(-Q-\mathrm{i}G/H^{\frac{1}{2}})\,\mathrm{d}x\right]; \\ (U+C_{g2})\dfrac{\sigma_2}{k_2}H^{-\frac{1}{2}}\exp\left[\displaystyle\int_0^x(-Q+\mathrm{i}G/H^{\frac{1}{2}})\,\mathrm{d}x\right]. \end{cases} \tag{4.15}$$

in which the constant $\frac{1}{2}\rho$ is omitted simultaneously from both k_1 and k_2 components. Expression (4.15) is rather complicated, but since for each component the right-hand side of (4.15) remains constant to the blockage point, we may calculate it at the convenient position $x = 0$, at which the integrals in (4.15) will vanish. We also note that though the wave action of the WKBJ solution is infinite at the blockage point $x = 0$, the action flux remains a finite limit there. Thus (4.15) can be written as

$$(U+C_g)\frac{E}{\sigma} = \begin{cases} \left[(U+C_{g1})\frac{\sigma_1}{k_1}H^{-\frac{1}{2}}\right]_{x=0}; \\ \left[(U+C_{g2})\frac{\sigma_2}{k_2}H^{-\frac{1}{2}}\right]_{x=0}. \end{cases}$$

In Appendix B, we further prove that

$$\left[(U+C_{g1})\frac{\sigma_1}{k_1}H^{-\frac{1}{2}}\right]_{x=0} = -\left[(U+C_{g2})\frac{\sigma_2}{k_2}H^{-\frac{1}{2}}\right]_{x=0}, \tag{4.16}$$

and we conclude that the action flux carried by the incident wave towards the blockage point equals that carried away by the reflected wave.

We are indebted to Dr Frank Henyey for pointing out that the above conclusion can be drawn directly from a simple and neat argument even without entering into the details of the solution. Henyey first defines

$$A = \tfrac{1}{2}\rho\,\mathrm{Im}\left(\gamma\eta^*\frac{\partial\eta}{\partial s}+\phi^*U\eta+\tfrac{1}{2}\mathrm{i}\phi^*\phi\right), \tag{4.17}$$

where asterisk denotes the complex conjugate. From (2.15) and (2.16), it immediately follows that

$$\frac{\partial}{\partial s}A = 0 \tag{4.18}$$

exactly. (Recall that (2.15) and (2.16) are themselves exact for linear short waves.) Henyey next points out that since

$$\phi \approx -\mathrm{i}\frac{\sigma}{k}\eta, \quad \frac{\partial\eta}{\partial s} \approx \mathrm{i}k\eta, \tag{4.19}$$

in the short-wavelength limit, the function A corresponds to the action flux. Therefore, from (4.18) and the fact that A is zero far into the forbidden region, it is zero everywhere, so the incoming and outgoing action fluxes are equal.

The approximations (4.19) are certainly insufficient for clarification of the content of A to the desired order. However, one may instead substitute the more accurate relations

$$\phi = -\mathrm{i}U\eta-\mathrm{i}\frac{g'}{n_0}\eta+\mathrm{i}\frac{\gamma}{n_0}\frac{\partial^2\eta}{\partial s^2}+\frac{U^2}{n_0}\frac{\partial\eta}{\partial s}+\frac{U\eta}{n_0}\frac{\partial U}{\partial s},$$

$$\frac{\partial\eta}{\partial s} \approx \mathrm{i}k\eta+\frac{1}{a}\frac{\partial a}{\partial s}\eta,$$

$$\frac{\partial^2\eta}{\partial s^2} \approx \mathrm{i}\frac{\partial k}{\partial s}\eta-k^2\eta+2\mathrm{i}k\frac{1}{a}\frac{\partial a}{\partial s}\eta.$$

Then, after a complicated manipulation, we still have

$$A = \left[\tfrac{1}{2}\rho(U+C_{g1})\frac{\sigma_1}{k_1}a_1^2\right]+\left[\tfrac{1}{2}\rho(U+C_{g2})\frac{\sigma_2}{k_2}a_2^2\right]+O\left(\frac{l^2}{L^2}\right), \tag{4.20}$$

in which the cancellation of the interaction terms between the k_1 and k_2 components is mainly due to (1.1) and (1.2) as well as (4.11). Thus the function A indeed corresponds to the net action flux of the WKBJ solution if both are calculated to the same order.

Incidently, we notice that since the value of A in the WKBJ solution remains negligible everywhere, it is obvious that the errors of such approximation, at least in amplitude, will not accumulate as $x \to \infty$.

5. Suppression of gravity–capillary waves

From the above discussion, it is clear that the blockage and reflection phenomena can occur to any scale of the water surface wave; the phenomena can be avoided only if the speed of the underlying large-scale flow is less than the minimum of C_g, which is about 18 cm/s.

For a very short gravity wave superimposed upon a longer one and propagating in the same direction, in the frame of reference moving with the longer wave its energy will be swept downstream and may be blocked on the forward slopes of the longer wave. In such a case, the very short gravity wave will be reflected as an extremely short capillary wave (see figure 1) and then be dissipated rapidly. Short gravity waves will eventually be erased entirely near the crests of the dominant wave as pointed out by Phillips (1981).

The influence of viscous dissipation can be estimated. When it is taken into account, the action conservation equation becomes (see Phillips 1977, §4.7)

$$\frac{\partial}{\partial x}\left[(U+C_g)\frac{E}{\sigma}\right] = -4\nu k^2 \frac{E}{\sigma},$$

or

$$\frac{1}{(U+C_g)E/\sigma}\frac{\partial}{\partial x}\left[(U+C_g)\frac{E}{\sigma}\right] = -\frac{4\nu k^2}{U+C_g}.$$

Integration yields

$$E \propto \left|\frac{\sigma}{U+C_g}\right| \exp\left[-\int \frac{4\nu k^2}{U+C_g}\,\mathrm{d}x\right]. \tag{5.1}$$

From (5.1) and (A 1), one can calculate the variation of the wave slope ak of a dissipating wave train. Figure 4 shows such variations of the incident and reflected waves for a specific case. In figure 4, the relative magnitudes of a_1 and a_2 near the blockage point are determined from the relation

$$(U+C_{g1})\frac{E_1}{\sigma_1} = -(U+C_{g2})\frac{E_2}{\sigma_2}, \tag{5.2}$$

because in the vicinity of the blockage point one may expect that the influence of the viscous dissipation is only slight for such a short distance and a moderate wavelength. However, since the meaning of the wave amplitude becomes vague in the vicinity of the blockage point, we do not imply in figure 4 that a and ak tend to infinity in this region according to (4.10).

For comparison, figure 4 also includes the variations of a and ak in the absence of viscous dissipation, showing that the reflected wave continually increases in steepness in a convergent region as it moves away from the blockage point, though its amplitude decreases. However, the viscosity will result in a faster and faster decline of a and ak of the reflected wave, in contrast to the incident wave in which the dissipation effect is almost imperceptible.

For a fixed dominant wave, longer wavelets will escape blockage; the shortest one that can continue to propagate past the crest has been estimated by Phillips (1981).

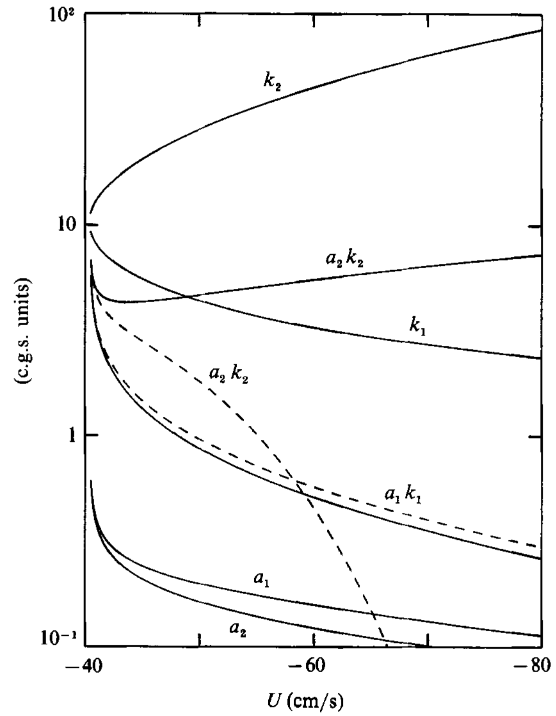

FIGURE 4. Variation of wavenumbers, amplitudes and slopes according to the WKBJ solution (4.9). A subscript 1 refers to the incident wave and subscript 2 to the reflected wave. The broken lines are the solutions including the viscous effect. The wavelength and slope of the underlying long wave are 25.13 cm and 0.4 respectively. The apparent frequency n_0 of the short wave is -120 rad/s.

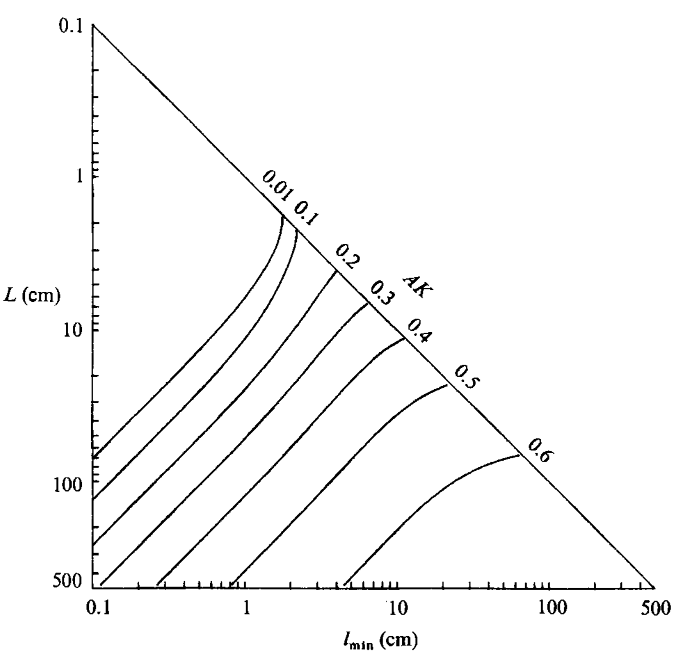

FIGURE 5. Minimum wavelength of the short wave escaping blockage as a function of the long-wave wavelength L and slope AK.

Here, figure 5 gives a numerical description of the minimum wavelength escaping blockage in terms of the characteristics of the dominant wave. In figure 5, the wavelength of the short wave is specified at the position a quarter of a wavelength from the crest of the long wave, almost at the mean water level. We also note that the calculations in figure 5, by using the solution of the long-wave field, are correct to second order in long-wave slope and take account of the effective gravitational acceleration. The results can therefore provide a guide for estimating the range of influence of the suppression process. A more accurate estimation can also be achieved by using a treatment due to Longuet-Higgins (1987) for precise calculations of the effective gravity g' and the long-wave field of finite amplitude. However, from figure 5 and Phillips' (1981) results, short waves with wavelengths of a few centimeters are likely to be erased by the successive gravity wave crests if the long-wave slope exceeds 0.2.

6. Gravity blockage phenomenon

In §4 we obtained the asymptotic solutions for a fairly general case, in which the waves can be gravity or capillary, and the underlying larger-scale flow can be any steady, irrotational currents (the results with a slight modification can even be applied to a uniform-vorticity flow as will be seen later). To apply the solutions to the phenomenon of gravity blockage, it is necessary only to specify a certain positive apparent frequency n_0 (see figure 6) and choose the right roots of equation (3.11) as the wavenumbers of the incident and reflected waves. Alternatively, one may derive *ab initio* the solutions for the gravity wave, which are much simpler than those for gravity–capillary waves.

When the surface-tension terms are neglected in the original third-order equation (2.19), we directly obtain

$$\frac{\partial^2\eta}{\partial x^2}+\left(-\mathrm{i}\frac{2n_0U+g'}{U^2}+\frac{3}{U}\frac{\partial U}{\partial x}\right)\frac{\partial\eta}{\partial x}-\left(\frac{n_0^2}{U^2}+2\mathrm{i}\frac{n_0}{U^2}\frac{\partial U}{\partial x}+\mathrm{i}\frac{1}{U^2}\frac{\partial g'}{\partial x}\right)\eta=0.$$

We may further simplify this by neglecting the derivative $\partial g'/\partial x$, for in the present case one may expect that $A/L \ll 1$. Consequently, we have

$$\frac{\partial^2\eta}{\partial x^2}+\left(-\mathrm{i}\frac{2n_0U+g'}{U^2}+\frac{3}{U}\frac{\partial U}{\partial x}\right)\frac{\partial\eta}{\partial x}-\left(\frac{n_0^2}{U^2}+2\mathrm{i}\frac{n_0}{U^2}\frac{\partial U}{\partial x}\right)\eta=0, \tag{6.1}$$

correct to $O((a/l)(Al/L^2))$.

The above equation can be rewritten in a form similar to (3.27) but with simpler expressions for the coefficients. In pure gravity waves, the dispersion relation is reduced to

$$n_0=(g'k)^{\frac{1}{2}}+Uk.$$

Expansion yields

$$k^2-\frac{2n_0U+g'}{U^2}k+\frac{n_0^2}{U^2}=0. \tag{6.2}$$

Thus (6.1) can also be written as

$$\frac{\partial^2\eta}{\partial x^2}+[-\mathrm{i}(k_1+k_2)+Q]\frac{\partial\eta}{\partial x}+[-k_1k_2+P]\eta=0, \tag{6.3}$$

with

$$Q=\frac{3}{U}\frac{\partial U}{\partial x},\quad P=-2\mathrm{i}\frac{n_0}{U^2}\frac{\partial U}{\partial x}. \tag{6.4}$$

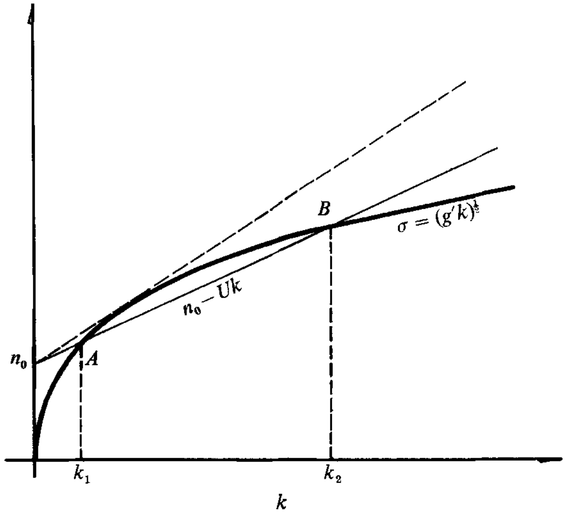

FIGURE 6. Solutions of the kinematical conservation equation analogous to figure 1, but for gravity waves and a positive n_0.

Equation (6.3) takes the exactly same form as (3.27), though its two coefficients Q and P are much simpler than (3.28) and (3.29). Incidentally, we notice that (3.27), derived from a considerably complicated procedure, virtually coincides with the result for the present simpler case, derived in a straightforward way. By change of dependent variable according to (4.1), equation (6.3) can also be reduced to the normal form

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+v(H+G) = 0, \tag{6.5}$$

where

$$H = \tfrac{1}{4}(k_1-k_2)^2, \quad G = \mathrm{i}\frac{1}{2}\frac{g'}{U^3}\frac{\partial U}{\partial x}. \tag{6.6}$$

The similarity between (6.5) and (4.2) is again remarkable. However, there exists a major difference between the capillary and gravity blockages that should be clarified before the results of §4 are applied to the present case.

In §4, we have assumed that the blockage point occurs downstream and the wave pattern is confined to the region of $x > 0$. This, in the event of capillary blockage, corresponds to the situation that the waves upon blockage have become even shorter to have greater group velocity to overcome the convection. Nevertheless, in the present case, the situation is the opposite one (the curve in figure 6 is convex towards positive ordinates while the relevant part in figure 1 is concave), so that the solutions in §4 are applicable only if the reflected wave is longer than the incident wave and the blockage occurs in a divergent region (in the previous case it occurs in a convergent zone). In the more important case when a longer gravity wave is reflected as a much shorter wave of the same type in a convergent region, the wave pattern will be confined to the region of $x < 0$. Consequently, the solution of (6.5) corresponding to (4.5) and (4.6) will become

$$v \approx A_0\,\mathrm{Ai}\,(-r)-C_0\,\mathrm{Ai}'\,(-r), \tag{6.7}$$

$$\left.\begin{aligned} &\text{where} \qquad \tfrac{2}{3}r^{\frac{3}{2}} = -\int_0^x H^{\frac{1}{2}}\,\mathrm{d}x, \\ &A_0 = \left(\frac{r}{H}\right)^{\frac{1}{4}} \cos\left(-\int_0^x \tfrac{1}{2}G/H^{\frac{1}{2}}\,\mathrm{d}x\right), \quad C_0 = r^{-\frac{1}{4}}H^{-\frac{1}{4}}\sin\left(-\int_0^x \tfrac{1}{2}G/H^{\frac{1}{2}}\,\mathrm{d}x\right). \end{aligned}\right\} \tag{6.8}$$

The adequacy of the above solution can again be verified by the same argument as described in §4. Alternatively, one may imagine that the x-axis is initially in the opposite direction (the values of H and G will remain unchanged in this exercise), so the expressions (4.5) and (4.6) will become available in this situation. Then an inverse of the x-axis will directly result in (6.7) and (6.8).

From the asymptotic expansions of Ai $(-r)$ and Ai$'$ $(-r)$, one may also obtain the corresponding WKBJ solution in this case:

$$\begin{aligned} \eta \approx\ & H^{-\frac{1}{4}}\exp\left[\int_0^x \tfrac{1}{2}(-Q+\mathrm{i}G/H^{\frac{1}{2}})\,\mathrm{d}x\right]\exp \mathrm{i}\left[\int_0^x k_1\,\mathrm{d}x - n_0 t + \tfrac{1}{4}\pi\right] \\ & + H^{-\frac{1}{4}}\exp\left[\int_0^x \tfrac{1}{2}(-Q-\mathrm{i}G/H^{\frac{1}{2}})\,\mathrm{d}x\right]\exp \mathrm{i}\left[\int_0^x k_2\,\mathrm{d}x - n_0 t - \tfrac{1}{4}\pi\right] \end{aligned} \tag{6.9}$$

for $x \ll 0$. It is interesting to notice that in (6.9) the phase discontinuity between the incident and reflected waves is opposite to that in (4.9).

An important feature of the above solutions is that they provide explicit formulae for computation of the wave profiles. Figure 7, similar to figure 4, illustrates the variations of a and ak of the incident and reflected waves for a specific case, showing that, because of the reduction in wavelength and increase in amplitude, the steepness of the reflected gravity wave increases significantly. The results in figure 7 are calculated from the action conservation equations (4.11) and (5.2) which are certainly fulfilled by the present solutions. However, in figure 8, we also calculate the instantaneous profile of the water surface as an example of the application of the formulae (6.7), (6.8) and (6.9). These calculations involve some numerical integration procedures as well as an evaluation of Ai $(-r)$ and Ai$'$ $(-r)$ by adding their power series term by term in appropriate number. The results clearly indicate that while the energy of the incident wave (long-dashed line) propagates upstream, the reflection process can very efficiently create choppiness on the sea surface downstream of the blockage point.

For application, it is also interesting to express explicitly the amplitude ratio a_1/a_2 of the incident wave to the reflected wave. From (4.10), (4.15) and (5.2), it follows that

$$\frac{a_1}{a_2} = \left[-\frac{(U+C_{g2})\,\sigma_2/k_2}{(U+C_{g1})\,\sigma_1/k_1}\right]^{\frac{1}{2}}.$$

Hence, for pure gravity waves, substituting the solutions of k_1 and k_2 derived from (6.2), we obtain

$$\frac{a_1}{a_2} = -1 - \frac{1}{2}\frac{g'}{n_0 U} + \frac{1}{2}\frac{g'}{n_0 U}(1+4n_0 U/g')^{\frac{1}{2}},$$

available in the region where the WKBJ solution is valid.

We finally note that though our previous discussion has assumed the underlying larger-scale flow to be irrotational, the present approach can be extended without difficulty to the case when the underlying flow has uniform vorticity. In this case, the velocity perturbation induced by the short waves remains irrotational, so that the

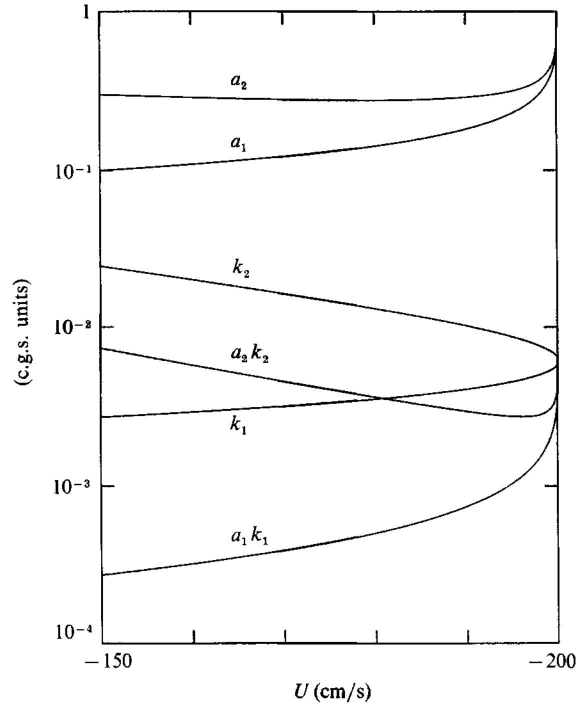

FIGURE 7. Variation of wavenumbers, amplitudes and slopes according to the WKBJ solution (6.9). A subscript 1 refers to the incident wave and subscript 2 to the reflected wave. The velocity distribution of the underlying current is $U = -2-0.01x$ (m/s) and the apparent frequency n_0 is 1.225 rad/s.

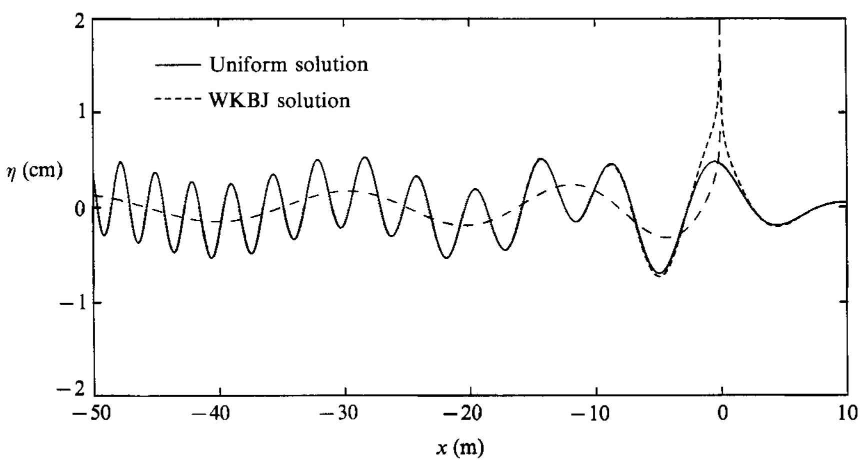

FIGURE 8. Two solutions of the instantaneous profile of the water surface under the same conditions as in figure 7. The long-dashed line represents the incident wave component in the WKBJ solution.

formulation in §2 is still valid, except that the dynamical boundary condition should now be derived directly from the momentum equation, instead of Bernoulli's equation, to include the effects of uniform vorticity. Recall that Bernoulli's equation is an integral of the momentum equation and the former in §2 has been differentiated with respect to s to combine with the kinematic boundary condition. Thus the existence of uniform vorticity in the mean flow affects the resulting third-order ordinary differential equation (2.19) only to within several extra terms in its coefficients. If these terms are included in the parameters $d_1, d_2, \ldots, d_5$ in (3.4) respectively, then our earlier solutions can still be applied to this case of uniform vorticity. From these solutions and the new expressions for the parameters d_j, one can similarly prove that even with strong uniform vorticity, wave action remains conserved everywhere including the blockage point, although the dispersion relation and the ratio of kinetic energy to potential energy of the linear short waves are influenced by vorticity, as pointed out by Teles da Silva & Peregrine (1988).

7. Conclusions

We have presented a method for a systematic investigation of the blockage phenomenon of surface waves by a larger-scale flow. The waves can be any linear gravity and/or capillary waves, and the underlying larger-scale flows can even possess uniform vorticity and a finite surface curvature; the latter may be induced by a longer wave of finite amplitude. The method first produces a second-order ordinary differential equation for the surface displacement η of the short wave, which in a rigorous two-scale approximation allows solutions that remain valid to the blockage point and to the region beyond, exhibiting decreasing exponential behaviour, so that a uniformly valid asymptotic solution can be derived straightforwardly.

Away from the blockage point, the uniform asymptotic solution can be replaced by its asymptotic expansion, which is the WKBJ solution but with specified relative amplitudes of the incident and reflected waves. From this solution, we clearly demonstrated that even with a curved surface and a variable effective gravity g', the action of the short wave is still conserved everywhere including the blockage point. We also note that the uniform vorticity in the larger-scale flow will significantly influence the dispersion relation and the relation between the short-wave amplitude and its energy (or action) density, as pointed out by Teles da Silva & Peregrine (1988); nevertheless the wave action remains conserved everywhere.

Gravity–capillary waves, blocked on the forward slopes of a steep, longer gravity wave, will be reflected as extremely short capillaries, then dissipated rapidly by viscosity. Hence those short wavelets with wavelengths of few centimeters will eventually be suppressed near the crests of long dominant waves even without breaking, as suggested by Phillips (1981). This phenomenon among others may significantly influence the fine structure of the sea surface.

In the blockage of gravity waves in a convergent zone, the subsequent reflection will result in a shortening of the wavelength and increase of the wave height. From figure 7, the difference in steepness between the incident and reflected waves at a fixed point is much greater than the modulations suffered by the incident wave in the region between this point and the blockage point. It is well known that waves are often broken in a river estuary or near a harbour entrance with an ebbing tide. However, from the present discussion, broken water will actually occur downstream of the blockage point, contrary to the earlier assumption that waves necessarily break at the blockage point.

Dr F. Henyey's comments and argument have helped us to correct a serious mistake in our theory, for which we are deeply grateful. The authors also wish to thank a referee of an earlier version of the manuscript for pointing out several important works on this problem, including Smith's (1975) paper. This research was supported by the Office of Naval Research under contract N00014-84K-0080.

Appendix A. Proof that the solution (4.9) fulfills the action conservation principle

The wave action density is

$$\frac{E}{\sigma} = \tfrac{1}{2}\rho \frac{\sigma}{k} a^2, \tag{A 1}$$

so that the differentiation of the action flux gives

$$\frac{\partial}{\partial x}\left[(U+C_g)\frac{E}{\sigma}\right] = \tfrac{1}{2}\rho\left\{\frac{\partial}{\partial x}\left[(U+C_g)\frac{\sigma}{k}\right] + (U+C_g)\frac{\sigma}{k}\frac{2}{a}\frac{\partial a}{\partial x}\right\}a^2. \tag{A 2}$$

From (4.3) and (4.10) we have

$$a^2 = \frac{2}{k_2-k_1}\exp\left[\int_0^x (-Q-\mathrm{i}G/H^{\frac{1}{2}})\,\mathrm{d}x\right]$$

for the k_1 component, and using (3.28), (4.3) and (4.4) we further have

$$-Q-\mathrm{i}G/H^{\frac{1}{2}} = \frac{1}{(k_1-k_2)(k_3-k_1)}[2d_4+2\mathrm{i}k_1 d_5+(3k_1-2k_2-k_3)\,k_1'-(k_3-k_1)\,k_2']. \tag{A 3}$$

Therefore, differentiation and reduction yield

$$\frac{2}{a_1}\frac{\partial a_1}{\partial x} = \frac{1}{(k_1-k_2)(k_3-k_1)}[2d_4+2\mathrm{i}k_1 d_5+(4k_1-2k_2-2k_3)\,k_1'].$$

Now substituting (3.4) into the above expression and employing (3.10) and (3.12) to eliminate k_2 and k_3, we have

$$\frac{2}{a_1}\frac{\partial a_1}{\partial x} = \frac{1}{(k_1-k_2)(k_3-k_1)}\left[4\frac{n_0}{\gamma}\frac{\partial U}{\partial x}+2\frac{1}{\gamma}\frac{\partial g'}{\partial x}-6\frac{Uk_1}{\gamma}\frac{\partial U}{\partial x}+\left(6k_1-2\frac{U^2}{\gamma}\right)k_1'\right], \tag{A 4}$$

and the denominator

$$(k_1-k_2)(k_3-k_1) = \frac{k_1}{\gamma}U^2-\frac{n_0^2}{\gamma k_1}-2k_1^2. \tag{A 5}$$

On the other hand, since $\quad U+C_g = U+\dfrac{1}{2}\dfrac{\sigma}{k}+\dfrac{\gamma k^2}{\sigma}$

and $$\sigma = n_0-Uk,$$

we have

$$(U+C_{g1})\frac{\sigma_1}{k_1} = -\tfrac{1}{2}U^2+\frac{1}{2}\frac{n_0^2}{k_1^2}+\gamma k_1. \tag{A 6}$$

The comparison between (A 5) and (A 6) also indicates that

$$(k_1-k_2)(k_3-k_1) = -2\frac{\sigma_1}{\gamma}(U+C_{g1}). \tag{A 7}$$

Therefore, substituting (A 4), (A 6) and (A 7) into (A 2), and carrying out the differentiation, we obtain

$$\frac{\partial}{\partial x}\left[(U+C_{g1})\frac{E_1}{\sigma_1}\right] = \tfrac{1}{2}\rho\left\{-2\frac{\sigma_1}{k_1}\frac{\partial U}{\partial x}-\frac{1}{k_1}\frac{\partial g'}{\partial x}-2(U+C_{g1})\frac{\sigma_1}{k_1^2}\frac{\partial k_1}{\partial x}\right\}a_1^2. \tag{A 8}$$

Equation (A 8) can be further reduced by expressing $\partial k_1/\partial x$ in terms of $\partial U/\partial x$ and $\partial g'/\partial x$. Since from (1.1) and (1.2),

$$n_0 = \sigma + kU = \text{const.}$$

and

$$\sigma = (g'k+\gamma k^3)^{\frac{1}{2}},$$

differentiation yields

$$0 = \frac{\partial \sigma}{\partial x}+U\frac{\partial k}{\partial x}+k\frac{\partial U}{\partial x}$$

$$= \frac{\partial \sigma}{\partial k}\frac{\partial k}{\partial x}+\frac{\partial \sigma}{\partial g'}\frac{\partial g'}{\partial x}+U\frac{\partial k}{\partial x}+k\frac{\partial U}{\partial x}$$

$$= (C_g+U)\frac{\partial k}{\partial x}+\frac{1}{2}\frac{k}{\sigma}\frac{\partial g'}{\partial x}+k\frac{\partial U}{\partial x}.$$

Thus we have

$$\frac{\partial k}{\partial x} = -\frac{k}{U+C_g}\frac{\partial U}{\partial x}-\frac{1}{2}\frac{1}{U+C_g}\frac{\mathrm{k}}{\sigma}\frac{\partial g'}{\partial x}. \tag{A 9}$$

Substituting (A 9) into (A 8), we finally obtain

$$\frac{\partial}{\partial x}\left[(U+C_{g1})\frac{E_1}{\sigma_1}\right] = 0. \tag{A 10}$$

A similar treatment may also lead to

$$\frac{\partial}{\partial x}\left[(U+c_{g2})\frac{E_2}{\sigma_2}\right] = 0. \tag{A 11}$$

This completes the proof.

Appendix B. Proof of (4.16)

First, it is useful to write the wavenumbers k_1 and k_2 in the forms

$$k_1 = M-N, \quad k_2 = M+N,$$

where $N = 0$ at the blockage point $x = 0$. Then

$$(U+C_{g1})\frac{\sigma_1}{k_1}H^{-\frac{1}{2}} = (U+C_{g1})\frac{\sigma_1}{k_1}\frac{2}{k_2-k_1}$$

$$= \left(\gamma k_1-\tfrac{1}{2}U^2+\frac{1}{2}\frac{n_0^2}{k_1^2}\right)\frac{1}{N}$$

$$= \frac{\gamma}{k_1}\left[k_1^2-\frac{1}{2}\frac{U^2}{\gamma}k_1+\tfrac{1}{2}k_2\left(\frac{U^2}{\gamma}-k_1-k_2\right)\right]\frac{1}{N}$$

$$= \frac{\gamma}{k_1}\left[\frac{U^2}{\gamma}-3M+N\right]$$

by virtue of (4.3), (A 6), (3.4), (3.10) and (3.12). The cancellation of N from the denominator ensures a finite limit as $x \rightarrow 0$. Consequently

$$\left[(U+C_{g1})\frac{\sigma_1}{k_1}H^{-\frac{1}{2}}\right]_{x=0} = \left[\frac{\gamma}{M}\left(\frac{U^2}{\gamma}-3M\right)\right]_{x=0}.$$

Similarly, we may obtain

$$\left[(U+C_{g2})\frac{\sigma_2}{k_2}H^{-\frac{1}{2}}\right]_{x=0} = -\left[\frac{\gamma}{M}\left(\frac{U^2}{\gamma}-3M\right)\right]_{x=0}.$$

Thus we prove that

$$\left[(U+C_{g1})\frac{\sigma_1}{k_1}H^{-\frac{1}{2}}\right]_{x=0} = -\left[(U+C_{g2})\frac{\sigma_2}{k_2}H^{-\frac{1}{2}}\right]_{x=0}.$$

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